Bipartite Graphs

Mentee: Himani Verma

Mentor: Feride Ceren Kose

Directed Reading Program

Semester: Fall 2020

Abstract

In this paper we will discuss a special type of graph – the bipartite graph. We will introduce relevant definitions, work through examples, prove a theorem, and then finally culminate our definitions and examples in the discussion of the application of the theorem. Basic knowledge of set theory is assumed.

Introduction

A *bipartite graph* can be defined as a graph whose vertices can be divided into two disjoint sets U and V such that every edge has end vertices in the each of the two sets. This would mean that for the graph G having the vertex set $V(G)$ and the edge set $E(G)$, vertices $u, v \in V(G)$ forming the edge $uv \in E(G)$, either $u \in U \text{ and } v \in V$, or $u \in V \text{ and } v \in U$. Sets U, V are the *partite sets*.

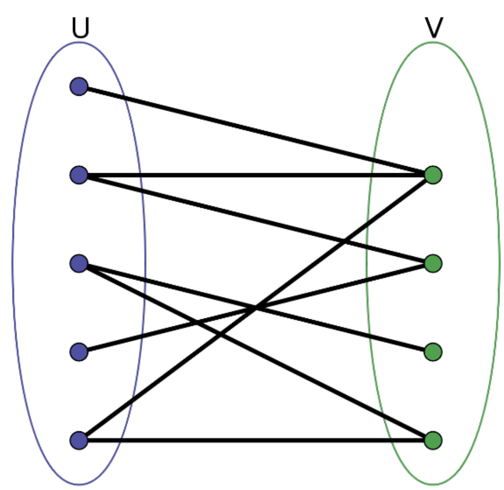

Fig 1. Bipartite Graph

Examples

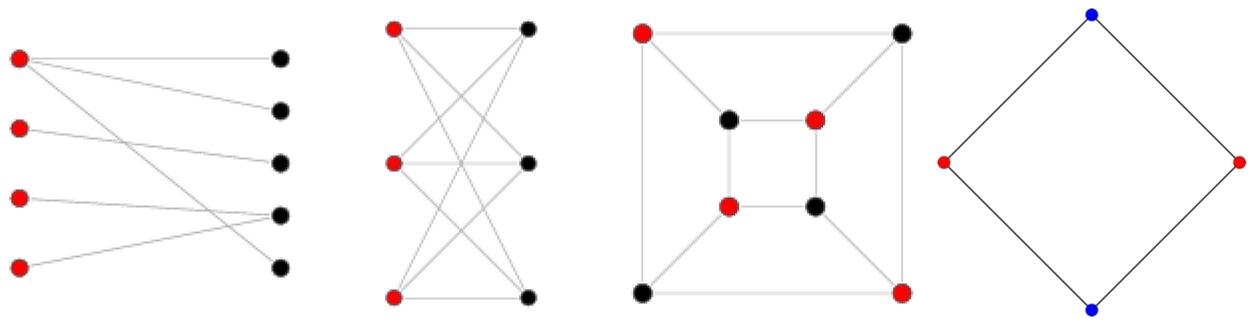

Fig 2. Examples of Bipartite Graphs

In the graphs above, all the graphs are bipartite. We can sort by the matching colors of vertices to their respective partite sets.

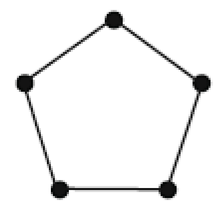

Fig 3. Non-bipartite graph

The graph above is non-bipartite since we cannot split the vertices of the graph into partite sets.

Characterization of Bipartite Graphs

Theorem: *A graph with at least two vertices is bipartite if and only if it contains no odd cycles.*

We will unwarp this theorem into two parts:

a) If a graph with at least two vertices is bipartite, then it contains no odd cycles.
b) If a graph with at least two vertices contains no odd cycles, then it is bipartite.

Before we begin, we should review a few basic definitions of graphs.

A *walk* is a sequence of vertices $v_1, v_2, v_3, \ldots, v_n$ such that each edge is $v_i\ v_{i+1} \in E(G)$ with values of i ranging from 1 to $n-1$. These vertices need not be necessarily distinct, but if they are distinct, the walk is then referred to as a *path*. If the walk has distinct edges, then the walk is called a trail. A *cycle* is defined as a trail where no other vertices are repeated apart from the start or the end vertex. Cycles can be *even* or *odd*, based on their number of edges. The total number of edges covered in a walk is also called the *length* of the walk. The *distance* between two vertices u and v is represented as $d(u, v)$ and it is the length of the shortest path connecting the two vertices.

Proof:

a) If a graph with at least two vertices is bipartite, then it contains no odd cycles.

Suppose graph G is bipartite.
Let X and Y be the partite sets of G such that $V(G) = X \cup Y$ and $X \cap Y = \emptyset$
For every $e \in E(G)$, $e = xy$ where $x \in X$ and $y \in Y$.

We will set up this problem as a proof by contradiction.
Starting point: Suppose not. That is, there is a bipartite graph G with at least one odd cycle.
To show: This supposition leads to a contradiction.
Let $C = (v_1, v_2, v_3, \ldots\ldots, v_n)$ where n is odd, and hence the odd cycle in G
$v_1 \in X$, Then $v_2 \in Y$.

$$v_i = \begin{cases} X & if\ i\ is\ odd \\ Y & if\ i\ is\ even \end{cases}$$

which implies, $v_n \in X$ since n is odd
$\Rightarrow v_1 v_n \in X$
$\Rightarrow v_1 v_n \in E(G)$, which contradicts our supposition that graph G is bipartite.
Therefore, graph G contains no odd cycles.

b) If a graph with at least two vertices contains no odd cycles, then it is bipartite.

Suppose graph G contains no odd cycles or all cycles in graphs in G are of even length.
Without loss of generality, we can assume that G is a connected graph.
Let $v \in V(G)$
$X = \{x \mid d(v, x)\ is\ even\}$
$Y = \{y \mid d(v, y)\ is\ odd\}$
Since G is connected, $V(G) = X \cup Y$ and $X \cap Y = \emptyset$
To show: for every edge $e \in E(G)$, $e = xy$ where $x \in X$ and $y \in Y$,
which can also be interpreted as $xy \notin E(G)$

Let $x, y \in X$.
Then, let $P: v \overset{*}{\rightarrow} x$ be a shortest path from v to x, and $Q: v \overset{*}{\rightarrow} y$ be a shortest path from v to y.
Since $x, y \in X$, P and Q are either both even or both odd.
Let w be the last common vertex of P and Q.

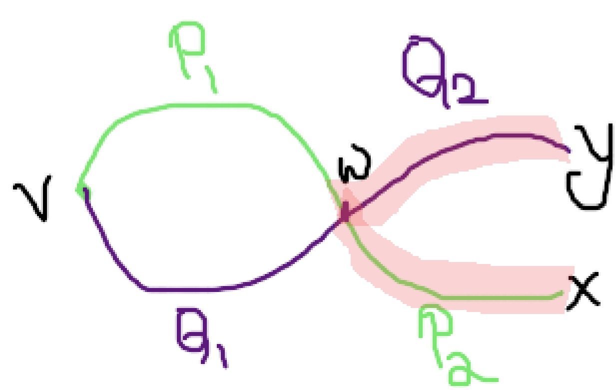

Fig 4.

Reading from the figure above,
$P = P_1 P_2$ and $Q = Q_1 Q_2$
$P_1 : v \overset{*}{\rightarrow} w$ and $P_2 : w \overset{*}{\rightarrow} x$
$Q_1 : v \overset{*}{\rightarrow} w$ and $Q_2 : w \overset{*}{\rightarrow} y$

Since P and Q are both the shortest paths, $|P_1| = |Q_1|$
$\Rightarrow$ $|P_2|$ and $|Q_2|$ have the same parity (either both are even, or both are odd).
The path highlighted is shown by:
$x \underset{P_2^{-1}}{\longrightarrow} w \underset{Q_2}{\rightarrow} y$
Or $P_2^{-1} Q_2$ is of even length.

If $xy \in E(G)$, then $P_2^{-1} Q_2$ and xy will form a cycle of an odd length.

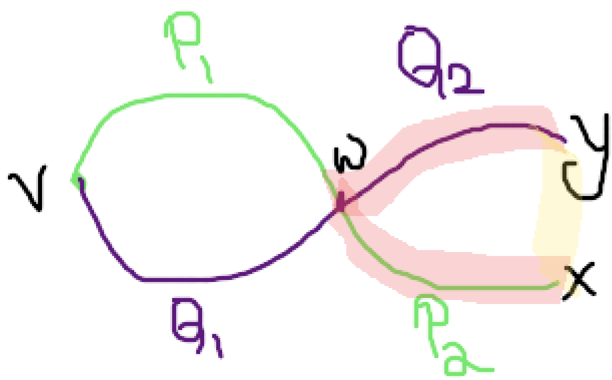

Fig 5.

But we our graph G contains no odd cycles, which is a contradiction.
Therefore, $xy \notin E(G)$.
The same holds true for $x, y \in Y$.
Therefore, graph G is bipartite.

From parts (a) and (b), a graph with at least two vertices is bipartite if and only if it contains no odd cycles.∎

Discussion

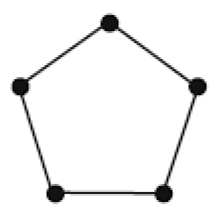

Fig 6. Non-bipartite graph

Following from the theorem above, the graph above has an odd cycle, therefore it is non-bipartite.

References

Harris, J., Hirst, J.L., and Mossinghoff, M. (2008). *Undergraduate Texts in Mathematics: Combinatorics and Graph Theory*. (S. Axler, and K.A. Ribet, Second Ed.). Springer.